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Improving Time-Optimal Maneuvers of Two-Link Robotic Manipulators

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I. Introduction

THIS Note discusses in detail numerical results of a parametric study for rest-to-rest time-optimal maneuvers of rigid two-link robotics manipulators. The problem of minimizing the time of planar maneuvers in terms of control and structure was considered parametrically. First, the time-optimal control strategy was applied. This strategy led to a bang-bang control in which the motors operated with the maximum torques changing directions at the switch time. The solutions were obtained by directly using Pontryagin's minimum principle (PMP). The analysis was then repeated for different lengths of particular links and for different torques applied at particular joints. The total length of the manipulator and the resultant torques generated by the shoulder and elbow motors were kept constant. For the numerical calculations, the set of data characterizing the IBM 7535 B 04 robot discussed in Refs. 1 and 2 was adapted.

II. Time-Optimal Control of Two-Link Robotic Manipulators

The equations derived from PMP and the corresponding boundary conditions form a two-point boundary value problem (TPBVP). Here we present the solutions generated by a numerical procedure that combines the forward-backward method with the shooting method to directly solve the TPBVP.[3] The procedure that is capable of determining the states, the costates, and the switching functions with a high numerical accuracy was discussed in more detail in Ref. 4. The procedure was used by the authors in Ref. 5 examine the effects of orientation of the plane of motion on the time-optimal maneuvers in the gravitational field. That work is extended here to include the effects of the links and the torque ratios.

Maneuvers of a two-link robotics manipulator are considered in plane y, z as shown in Fig. 1. Mass moments of inertia of the links with respect to their centers of mass, located at l_{c1} and l_{c2}, respectively, are I_1 and I_2. The states are $x_1 = \varphi_1$, $x_2 = \dot{\varphi}_1$, $x_3 = \varphi_2$, and $x_4 = \dot{\varphi}_2$. The manipulator is driven by motors installed at the shoulder and elbow joints and generating torques u_1 and u_2, respectively.

In terms of the states x_i, $i = 1, \ldots, 4$, the equation of motion of the manipulator can be obtained in the form

$$\dot{x}(t) = \boldsymbol{A}(x) + C(x)u(t) \tag{1}$$

where $\boldsymbol{A}$ and C are the vector and the matrix of nonlinear functions of states (see Ref. 4), and $\boldsymbol{u}$ is the vector of controls, represented here by torques $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. The control torques are bounded as

$$U_i^- \le \boldsymbol{u}_i(t) \le U_i^+ \tag{2}$$

For the time-optimal control problem, the state departing from the initial conditions, $x(0) = x_0$, must reach the final conditions,

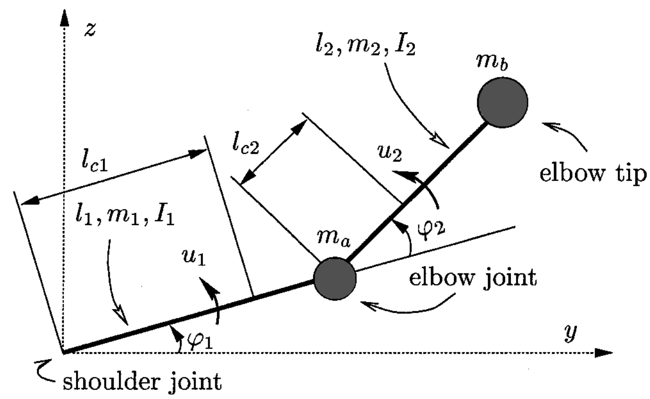

Fig. 1 Planar two-link robotics manipulator.

$x(t_f) = x_f$, in a minimum time that is $t_f \rightarrow$ min. After introducing the costates $p(t)$ and applying Pontryagin's Minimum Principle, the optimal solution must satisfy the necessary conditions

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x} \tag{3}$$

where the Hamiltonian $H(x, u, p) = 1 + p^T[A(x) + C(x)u(t)] \xrightarrow{u}$ min. The control torques have the form of a bang-bang control

$$u_i = \begin{bmatrix} U_i^+ & \text{for} & G_i < 0 \\ U_i^- & \text{for} & G_i > 0 \end{bmatrix} \tag{4}$$

where $G_i = p_j C_{ji}$ is the switch function corresponding to the control u_i, and C_{ji} are the components of ith column of matrix C. For the time-optimal trajectory, the Hamiltonian must satisfy the condition that $H(x, u, p) = 0$ for all t.

The solution of four state and four costate equations must satisfy eight initial and final conditions imposed on the state only. This TPBVP is solved numerically for assumed values of x_0, x_f, U_i^-, and U_i^+ using the procedure discussed in Refs. 3 and 4. Any solution that meets PMP for the manipulator as defined earlier is referred to as time-optimal. Here we concentrate on discussing some of these solutions.

III. Simulation Results

In the analysis presented, it is assumed that the links are made of cylindrical uniform bars of diameters d_i for which $I_i = m_i(l_i^2/12 + d_i^2/16)$. Additionally, for the purpose of this analysis, it is assumed that the total length of the manipulator $L_t = l_1 + l_2$ and the total torque $U_t = U_1 + U_2$ are constant. The calculations are performed for different length and different torque ratios defined by

$$R_L = l_1/l_2 \qquad R_U = U_1/U_2 \tag{5}$$

If $L_t = 0.65$ m, $R_L = 1.6$ and $U_t = 34$ Nm, $R_U = 2.7778$, and for $d_1 = d_2 = 0.10987$ m the following parameters are identical to those given in Refs. 1 and 2 for the IBM 7535 B 04 robot:

$$l_1 = 2l_{c1} = 0.4\ \text{m} \qquad U_1^{\mp} = \mp 25\ \text{Nm} \qquad I_1 = 0.4167393\ \text{kg}\cdot\text{m}^2$$

$$l_2 = 2l_{c2} = 0.25\ \text{m} \qquad U_2^{\mp} = \mp 9\ \text{Nm} \qquad I_2 = 0.1102435\ \text{kg}\cdot\text{m}^2$$

$$m_1 = 29.58\ \text{kg} \qquad m_2 = 18.49\ \text{kg} \tag{6}$$

In terms of the states, the initial and the final conditions for the rest-to-rest maneuvers from straight-to-straight configurations are given as $x(0) = [0.0\ 0.0\ 0.0\ 0.0]^T$, $x(t_f) = [\varphi_{1_f}\ 0.0\ 0.0\ 0.0]^T$, where φ_{1_f} is final maneuver angle.

A. Effects of the Length Ratio

Theoretically, the manipulator can access any point within the circle of radius L_t only if $R_L = 1$. Therefore, in order to improve accessibility, the length ratio for the robot analyzed in the previous

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Table 1 Physical parameters and optimal t_f for φ_{1_f} = 0.76 and R_U = 2.7778

R_L	l_1	l_2	m_1	m_2	I_1	I_2	t_f
1.00	0.325	0.325	24.04	24.04	0.22969	0.22969	0.896209
1.05	0.333	0.317	24.62	23.45	0.24599	0.21415	0.889071
1.10	0.340	0.310	25.18	22.89	0.26224	0.20002	0.881879
1.20	0.355	0.295	26.22	21.85	0.29444	0.17543	0.868256
1.30	0.367	0.283	27.17	20.89	0.32611	0.15487	0.856113
1.40	0.379	0.271	28.04	20.03	0.35710	0.13754	0.845438
1.50	0.390	0.260	28.84	19.23	0.38733	0.12283	0.836022
1.60	0.400	0.250	29.58	18.49	0.41674	0.11024	0.827643

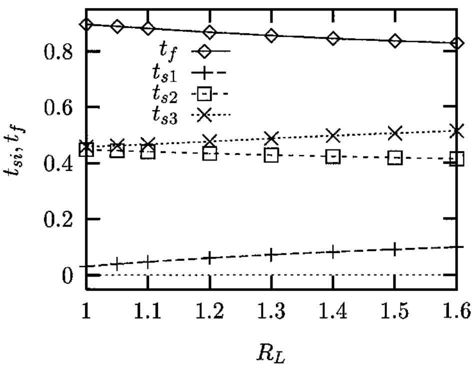

Fig. 2 Effects of R_L on optimal switch times and final time for φ_{1_f} = 0.76 and R_U = 2.7778.

section is gradually reduced from 1.6 to 1.0 while keeping the maneuver angle $\varphi_{1_f} = \varphi_1(t_f) = 0.76$ rad and keeping the torque ratio $R_U = 2.7778$ constant. As shown in Fig. 2, the maneuver time was increased from $t_f = 0.827643$ s for $R_L = 1.6$ to $t_f = 0.896209$ for $R_L = 1.0$. Figure 2 indicates that the design for minimum maneuver time, for which the shoulder link should be longer than the elbow link, is in conflict with the design for accessibility, for which the difference between the two links should be minimum. Table 1 lists the final time and the values of the related physical parameters of the manipulator when R_L changes. The optimal trajectory of the tip of manipulator for $R_L = 1.0$ becomes more circular and shorter in comparison with the trajectory for $R_L = 1.6$. However the latter trajectory requires about 8% more maneuver time. The faster traveling time for $R_L = 1.6$ may be attributed to the fact that the second link, due to its inclination from the first link, is in a shorter distance from the shoulder joint during most of the maneuver. This shorter distance of the second link from the shoulder lowers the total mass moment of inertia of the whole manipulator and permits its overall faster rotation.

B. Effects of the Torque Ratio

Using the torque ratio in Eq. (5), the shoulder and the elbow torque limits vary according to

$$U_1 = U_t[R_U/(1 + R_U)] \qquad U_2 = U_t[1/(1 + R_U)]$$

where $U_t = 34$ Nm.

Here the torque ratio varies within $0.78947 \leq R_U \leq 3.0$, resulting in the torques u_1 and u_2 as listed in Table 2 (in Sec. III.A this ratio was $R_U = 2.7778$), while the link ratio $R_L = 1.0$ is kept constant. The initial and the final conditions are the same as before, with $\varphi_{1_f} = 0.76$ rad.

Figure 3 shows the effect of R_U (change in the distribution of control torques applied to the manipulator) on the optimal maneuver time t_f and the switch times t_{si}. As R_U increases, final time decreases to a minimum $t_f = 0.859362$ at $R_U = 1.3448$ (see Table 2). It indicates that there is an optimal ratio at which the maneuver time is the fastest. The trajectory for $R_U = 1.3448$, despite being longer, provides about 4% faster maneuver time.

Table 2 Control torques and optimal t_f for φ_{1_f} = 0.76 and R_L = 1.0

R_U	U_1	U_2	t_f
0.78947	15.0	19.0	0.8724345
0.88889	16.0	18.0	0.8680668
1.00000	17.0	17.0	0.8642263
1.12500	18.0	16.0	0.8613008
1.26667	19.0	15.0	0.8597692
1.3448	**19.5**	**14.5**	**0.859362**
1.42857	20.0	14.0	0.8599207
1.61538	21.0	13.0	0.8621044
1.83333	22.0	12.0	0.8664952
2.09091	23.0	11.0	0.8739482
2.40000	24.0	10.0	0.8847785
2.7778	**25.0**	**9.0**	**0.896209**
3.00000	25.5	8.50	0.8978919

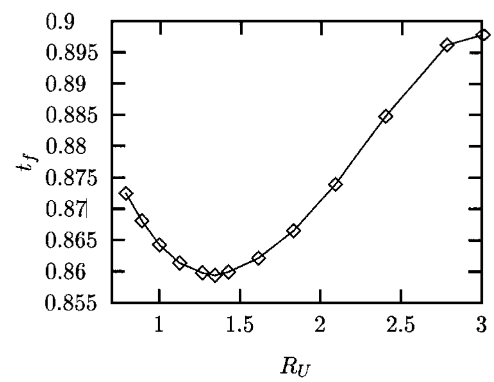

Fig. 3 Effects of R_U on optimal final time for φ_{1_f} = 0.76 and R_L = 1.0.

IV. Conclusions

Performance of two-link manipulators for time-optimal maneuvers can be improved by varying some of their structural parameters. Sensitivity of the maneuver time to the structural parameters considered appears to be generally low. For example, varying the torque ratio by about 380% or varying the length ratio by 60% changes the maneuver time by only about 4 and 8%, respectively. On the other hand, the control parameters (i.e., the switch times) vary very little even for a relatively wide range of structural parameters. Also, the number of switches was found to be predominantly three.

A faster maneuver time could be obtained by increasing the ratio of the length of shoulder link over the length of elbow link. This increase however, would reduce the accessibility of the manipulator. For a fixed geometry of the links, one can reduce the maneuver time by altering the torques applied at the shoulder and at the elbow joints while keeping the total torque constant. It appears to be an optimal ratio of these torques at which the time of optimal maneuver is the fastest.

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